

Open Book Structures on Moment-Angle Manifolds $Z^{\mathbb{C}}(\Lambda)$ and Higher Dimensional Contact Manifolds.*

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Abstract

We construct open book structures on moment-angle manifolds and give a new construction of examples of contact manifolds in arbitrarily large dimensions.

Key Words: Open book decomposition, moment-angle manifolds, contact structures.

1 Introduction

The topology of generic intersections of quadrics in \mathbb{R}^n of the form:

$$\sum_{i=1}^n \lambda_i x_i^2 = 0, \quad \sum_{i=1}^n x_i^2 = 1,$$

where $\lambda_i \in \mathbb{R}^k$, $i = 1, \dots, n$ has been studied for many years: For $k = 2$ they are diffeomorphic to a triple product of spheres or to the connected sum of sphere products ([16], [15]); for $k > 2$ this is no longer the case ([2], [3]) but there are large families of them which are again connected sums of spheres products ([13]).

Let $\Lambda = (\lambda_1, \dots, \lambda_n)$. The generic condition, known as *weak hyperbolicity* and equivalent to regularity, is the following:

If $J \subset 1, \dots, m$ has k or fewer elements then the origin is not in the convex hull of the λ_i with $i \in J$.

A crucial aspect of these varieties is that they admit natural group actions. All of them admit \mathbb{Z}_2^n actions. Their complex versions in \mathbb{C}^n

*Partially supported by project PAPIIT-DGAPA IN100811 and IN108112 and project CONACyT 129280.

$$\sum_{i=1}^n \lambda_i |z_i|^2 = 0, \quad \sum_{i=1}^n |z_i|^2 = 1,$$

(now known as *moment angle manifolds*) admit natural \mathbb{T}^n actions. The quotient is in both cases a polytope \mathcal{P} that determines completely the manifolds and the actions. We will use both notations $Z = Z(\mathbf{\Lambda}) = Z(\mathcal{P})$ and $Z^c = Z^c(\mathbf{\Lambda}) = Z^c(\mathcal{P})$ for these manifolds¹. They both fall under the general concept of *generalized moment angle complexes* ([5] and [2]).

An open book construction with these manifolds was used to provide a description of the topology of some cases not covered by the main theorem in [16] (see remark on page 281). In [13] it is a principal technique for proving the results about some families when $k > 2$.

In section 2 we recall this construction, underlining its consequences for moment-angle manifolds:

If \mathcal{P} is a simple convex polytope and F is one of its facets, there is an open book decomposition of $Z^c(\mathcal{P})$ with binding $Z^c(F)$ and trivial monodromy.

When $k = 2$, we also give a topological description of the binding and leaves of the decomposition, in terms of the *odd cyclic partitions* of n that classify the varieties². This description is almost complete in the case of moment-angle manifolds for $k = 2$: The leaf is the interior of a manifold that can be:

- a) a product $\mathbb{S}^{2n_2-1} \times \mathbb{S}^{2n_3-1} \times \mathbb{D}^{2n_1-2}$,
- b) a connected sum along the boundary of products of the form $\mathbb{S}^p \times \mathbb{D}^{2n-p-4}$,
- c) in some cases, in the connected sum there may appear summands of the form $\mathbb{S}^{2p-1} \times \mathbb{S}^{2n-2p-3} \setminus \mathbb{D}^{2n-4}$ or of the exterior of an embedded $\mathbb{S}^{2q-1} \times \mathbb{S}^{2r-1}$ in \mathbb{S}^{2n-4} .

The specific products that appear in the above formulae will be described precisely in 3 in section 3. The proofs will be postponed to section 7 where they will follow from a more general theorem for the corresponding real manifolds that requires, as in [16], additional dimensional and connectivity hypotheses. The corresponding result in the real case amounts to the topological description of the *half* manifolds $Z_+ = Z \cap \{x_1 \geq 0\}$, complementing the results in [16]. Parts of these theorems follow directly from the results in [16], other parts require

¹The facets of the polytope correspond with the intersections of the manifolds with the coordinate hyperplanes. Technically, we would have to add the datum n to the notations $Z(\mathcal{P})$ and $Z^c(\mathcal{P})$ to specify that some *facets* may be empty. In Theorem 1 this would mean that the binding might be empty. In all our examples this will be explicit.

²See section 7.

the use of many of the technical lemmas proved there. All these manifolds with boundary are also generalized moment-angle complexes.

In section 4 we recall some results in [21] which state that the **LV-M** complex manifolds $\mathcal{N}(\mathbf{\Lambda})$ which correspond to configurations $\mathbf{\Lambda}$ which satisfy an arithmetic condition (Condition **(K)**) fibre (possibly as Seifert fibrations) over toric varieties (or orbifolds with simple singularities) with fibre a complex torus. From this we conclude that the pages of the open books described in section 2 are complex manifolds of the form $\mathcal{N}(\mathbf{\Lambda}) - \mathcal{N}(\mathbf{\Lambda}_0)$ where $\mathbf{\Lambda}_0$ is obtained from $\mathbf{\Lambda}$ by suppressing one element of the configuration.

In section 5 we show that every moment-angle manifold $Z^c(\mathbf{\Lambda})$ admits an almost contact structure and, as a consequence, these manifolds also admit a quasi-contact structure. We recall that there is the conjecture [18] that every almost contact manifold is in fact a contact manifold.

In section 6 we describe a new construction of compact contact manifolds $Z^c_{(\mathbf{\Lambda}, n, s)}$ in arbitrarily large odd dimensions. These manifolds are generalizations of the moment-angle manifolds $Z^c(\mathbf{\Lambda})$ that have been studied by V. Gómez Gutiérrez and the second author [15].

Even dimensional moment-angle manifolds can be given complex structures but admit symplectic structures only in a few well-known cases [20] so it is surprising that many of the odd dimensional ones admit contact structures. In this respect we remark that using results of C. Meckert [19] and Y. Eliashberg [8] it is possible to give a contact structure on these manifolds. See remark 4 in section 6.

However our method is distinct and in some sense explicit without using open books. What we use is the heat flow method described in [1]. In other words, we construct a positive confoliation α which is conductive i.e. every point of the manifold can be connected by a suitable Legendrian curve to a point where the form is positive (see definition 4 in 6.3 in section 6).

In section 7 we recall some old results about the topology of intersections of quadrics and use them to proof a new one from which theorem 3 of section 2 follows.

2 Open book decomposition

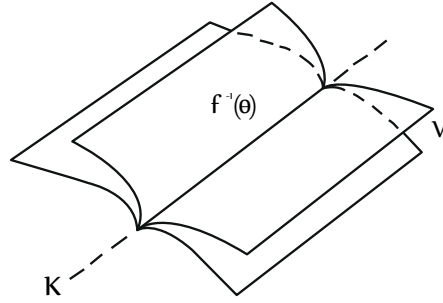
Let us recall the concept of an *open book*, introduced by H. E. Winkelnkemper in 1973 (see [26]). An open book in a smooth manifold V is a pair (\mathcal{K}, f) consisting of the following:

- a) A proper submanifold \mathcal{K} of codimension two in V with trivial normal

bundle, so \mathcal{K} admits a neighborhood \mathbf{N} diffeomorphic to $\mathbb{D}^2 \times \mathcal{K}$.

- b) A locally trivial smooth fibration $f : V \setminus \mathcal{K} \rightarrow \mathbb{S}^1$ such that there exists a neighborhood \mathbf{N} of \mathcal{K} as above, in which f is the normal angular coordinate.

The submanifold \mathcal{K} is called the *binding* and the fibres of f are the *pages* of the open book (\mathcal{K}, f) .



Also we can define an open book as follows: Let F be a manifold with a boundary and ϕ a self diffeomorphism of F that is the identity on ∂F . The mapping torus $\Sigma(F, \phi)$ of ϕ , i.e., the quotient of $F \times \mathbb{R}$ by the equivalence relation generated by $(p, t) \sim (\phi(p), t + 1)$, is a manifold with boundary $\partial \Sigma(F, \phi) = \partial F \times \mathbb{R}/\mathbb{Z}$. Collapsing each circle $\{p\} \times \mathbb{R}/\mathbb{Z}$, to a point, $\{p\} \in \partial F$ we obtain a manifold $\bar{\Sigma}(F, \phi)$ without boundary called the *relative mapping torus* of ϕ .

This manifold has an obvious open book structure whose binding is a copy of ∂F , the collapsed $\partial F \times \mathbb{R}/\mathbb{Z}$, and whose mapping to the circle is induced by the projection $F \times \mathbb{R} \rightarrow \mathbb{R}$ and so the pages are copies of F .

We call *monodromy* of an open book (\mathcal{K}, f) in a manifold V to any self diffeomorphism of a page F , such that there is a diffeomorphism $\bar{\Sigma}(F, \phi) \rightarrow V$ taking the obvious open book to (\mathcal{K}, f) .

Passing to our varieties, Z admits a \mathbb{Z}_2^n action obtained by changing the signs of the coordinates. The quotient is a simple polytope \mathcal{P} which can be identified with the intersection of Z and the first orthant of \mathbb{R}^n . It follows that Z can be reconstructed from this intersection by reflecting it on all the coordinate hyperplanes.

By a simple change of coordinates $r_i = x_i^2$, this quotient can be identified with the d -dimensional convex polytope given by

$$\sum_{i=1}^n \lambda_i r_i = 0, \quad \sum_{i=1}^n r_i = 1, \\ r_i \geq 0.$$

Let Λ' be obtained from Λ by adding an extra λ_1 which we interpret as the coefficient of a new extra variable x_0 , so we get the manifold Z' :

$$\lambda_1 (x_0^2 + x_1^2) + \sum_{i>1} \lambda_i x_i^2 = 0, \\ x_0^2 + x_1^2 + \sum_{i>1} x_i^2 = 1.$$

Let Z_+ be the intersection of Z with the half space $x_1 \geq 0$ and Z'_+ the intersection of Z' with the half space $x_0 \geq 0$.

The boundary of Z'_+ is Z (this shows that Z is always the boundary of a parallelizable manifold).

Z_+ admits an action of \mathbb{Z}_2^{n-1} by changing signs on all the variables except x_1 and the quotient is again \mathcal{P} . In other words, Z_+ can be obtained from $\mathcal{P} \times \mathbb{Z}_2^{n-1}$ by reflecting \mathcal{P} on all the coordinate hyperplanes except $x_1 = 0$.

Consider also the manifold Z_0 which is the intersection of Z with the subspace $x_1 = 0$. Z_0 is the boundary of Z_+ .

\mathbb{S}^1 acts on Z' (rotating the coordinates (x_0, x_1)) with fixed set Z_0 . Its quotient can be identified with Z_+ . The map

$$(x_0, x_1, x_2, \dots, x_m) \mapsto \left(\sqrt{x_0^2 + x_1^2}, x_2, \dots, x_m \right)$$

is a retraction from Z' to Z_+ which restricts to the retraction from Z to Z_+

$$(x_1, x_2, \dots, x_m) \mapsto (|x_1|, x_2, \dots, x_m).$$

Observe further that this retraction restricted to Z'_+ is homotopic to the identity: the homotopy preserves the coordinates x_i , $i \geq 2$ and folds gradually the half space $x_0 \geq 0$ of the x_0, x_1 plane into the ray $x_0 = 0$, $x_1 \geq 0$ preserving the distance to the origin.

So Z is the double of Z_+ and Z' is the double of Z'_+ , and Z' is the open book with binding Z_0 , page Z_+ and trivial monodromy.

Theorem 1. *Every manifold Z' is an open book with trivial monodromy whose binding is Z_0 and page Z_+ .*

Since the manifold $Z^\mathbb{C}(\mathcal{P})$ can be considered for each i as a manifold Z' with repeated coefficient λ_i , then it is an open book with binding the manifold obtained by taking $z_i = 0$. This proves:

Theorem 2. *If \mathcal{P} is a simple convex polytope and F is one of its facets, there is an open book decomposition of $Z^\mathbb{C}(\mathcal{P})$ with binding $Z^\mathbb{C}(F)$ and trivial monodromy.*

When $k = 2$ the topology of $Z^\mathbb{C}$ can be described precisely (see [16] and section 8) and that includes also that of all the bindings. It can be expressed in terms of the *cyclic partition* associated to $Z^\mathbb{C}$. We have a precise description of the topology of the leaves in most cases. For our purposes we will only need the case where the total manifold is a moment-angle manifold which is described in the following:

Theorem 3. *Let $k = 2$, and consider the manifold $Z^\mathbb{C}$ corresponding to the cyclic partition $n = n_1 + \dots + n_{2\ell+1}$. Consider the open book decomposition of $Z^\mathbb{C}$ corresponding to the binding at $z_1 = 0$, as given by Theorem 2. Then the leaf of this decomposition is diffeomorphic to the interior of:*

a) If $\ell = 1$, the product

$$\mathbb{S}^{2n_2-1} \times \mathbb{S}^{2n_3-1} \times \mathbb{D}^{2n_1-2}.$$

b) If $\ell > 1$ and $n_1 > 1$, the connected sum along the boundary of $2\ell + 1$ manifolds:

$$\coprod_{i=2}^{\ell+2} (\mathbb{S}^{2d_i-1} \times \mathbb{D}^{2n-2d_i-3}) \coprod \coprod_{i=\ell+3}^1 (\mathbb{D}^{2d_i-2} \times \mathbb{S}^{2n-2d_i-2}).$$

c) If $n_1 = 1$ and $\ell > 2$, the connected sum along the boundary of 2ℓ manifolds:

$$\coprod_{i=3}^{\ell+1} (\mathbb{S}^{2d_i-1} \times \mathbb{D}^{2n-2d_i-3}) \coprod \coprod_{i=\ell+3}^1 (\mathbb{D}^{2d_i-2} \times \mathbb{S}^{2n-2d_i-2})$$

$$\coprod (\mathbb{S}^{2d_2-1} \times \mathbb{S}^{2d_{\ell+2}-1} \setminus \mathbb{D}^{2n-4}).$$

d) If $n = 1$ and $\ell = 2$, a connected sum along the boundary of two manifolds:

$$(\mathbb{S}^{2d_2-1} \times \mathbb{S}^{2d_4-1} \setminus \mathbb{D}^{2n-4}) \coprod \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the exterior of an embedded $\mathbb{S}^{2n_2-1} \times \mathbb{S}^{2n_5-1}$ in \mathbb{S}^{2n-4} .

The proof of this theorem and of its real version will appear in section 7. Similar results can be given when $k > 2$ for large families in the spirit of [13].

3 Examples of open book decompositions of some moment-angle manifolds.

To illustrate the variety of decompositions obtained we give now some examples, by direct application of the previous theorem (which gives always the decomposition with binding at $z_1 = 0$). The reader may get a better feeling if she or he looks at these examples in the light of the proof of the theorem in section 7. An additional feature of these examples is that we obtain three different open book decompositions of the same moment-angle manifold:

Let $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. The 5-tuple in \mathbb{C} corresponding to the five roots of unity ($\mathbf{\Lambda}$ satisfies the weak hiperbolicity condition).

1. Consider $\mathbf{\Lambda}''$ where λ_1 has multiplicity 3. We obtain the moment-angle manifold

$$Z^c(\mathbf{\Lambda}) = \#_2 \mathbb{S}^7 \times \mathbb{S}^4 \#_3 \mathbb{S}^3 \times \mathbb{S}^8.$$

When $z_1 = 0$ we have a configuration $\mathbf{\Lambda}'$, where now the multiplicity of λ_1 is two. Then the binding is

$$Z_o^c(\mathbf{\Lambda}') = \#_2 \mathbb{S}^5 \times \mathbb{S}^4 \#_3 \mathbb{S}^3 \times \mathbb{S}^6.$$

The page, by Theorem 3(b), is in this case the manifold

$$Z_+^c(\mathbf{\Lambda}) = \left(\coprod_3 \mathbb{S}^3 \times \mathbb{D}^7 \right) \amalg \left(\coprod_2 \mathbb{D}^6 \times \mathbb{S}^4 \right).$$

2. Consider $\mathbf{\Lambda}''$ where now λ_2 has multiplicity 3. The moment-angle manifold is the same:

$$Z^c(\mathbf{\Lambda}) = \#_2 \mathbb{S}^7 \times \mathbb{S}^4 \#_3 \mathbb{S}^3 \times \mathbb{S}^8.$$

When $z_1 = 0$ the binding is,

$$Z_o^c(\mathbf{\Lambda}') = \mathbb{S}^5 \times \mathbb{S}^3 \times \mathbb{S}^1.$$

and the page, by Theorem 3(d), is

$$Z_+^c(\mathbf{\Lambda}) = \mathbb{S}^7 \times \mathbb{S}^3 \setminus \mathbb{D}^{10} \amalg \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the complement of an embedding $\mathbb{S}^5 \times \mathbb{S}^1$ in \mathbb{S}^{10} .

3. Consider now $\mathbf{\Lambda}''$ where now λ_3 has multiplicity 3. The moment-angle manifold is again

$$Z^c(\mathbf{\Lambda}) = \#_3 \mathbb{S}^3 \times \mathbb{S}^8 \#_2 \mathbb{S}^7 \times \mathbb{S}^4.$$

When $z_1 = 0$ the binding is

$$Z_0^c(\mathbf{A}') = \mathbb{S}^1 \times \mathbb{S}^7 \times \mathbb{S}^1.$$

and the page, by Theorem 3(d), is

$$Z_+^c(\mathbf{A}) = \mathbb{S}^7 \times \mathbb{S}^3 \setminus \mathbb{D}^{10} \coprod \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the complement of the (unique, up to isotopy) embedding of $\mathbb{S}^1 \times \mathbb{S}^1$ in \mathbb{S}^{10} .

When we take λ_4 or λ_5 with multiplicity 3 we obtain, by symmetry the same open book decompositions as in examples 3 and 2 above, respectively.

Let: $\mathbf{\Lambda}$ be an admissible configuration,
 $Z^c(\mathbf{\Lambda})$ the moment-angle manifold corresponding to $\mathbf{\Lambda}$,
 $Z_+^c(\mathbf{\Lambda})$ the page and $Z_0^c(\mathbf{\Lambda})$ the binding of the open book decomposition.

| Admissible Configuration | Moment-Angle Manifold | Binding | Page |
|---|---|---|---|
| $\mathbf{\Lambda} = (1, 1, i, -1 - i)$ | $Z^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{S}^3$ | $Z_0^c(\mathbf{\Lambda}) = \mathbb{T}^3$ | $Z_+^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{D}^2$ |
| $\mathbf{\Lambda} = (1^{n-2}, i, -1 - i)$ | $Z^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{S}^{2n-5}$ | $Z_0^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{S}^{2n-7}$ | $Z_+^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{D}^{2n-6}$ |
| $\mathbf{\Lambda} = (i, i, 1, -1 - i, -1 - i)$ | $Z^c(\mathbf{\Lambda}) = \mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{S}^3$ | $Z_0^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{S}^3$ | $Z_+^c(\mathbf{\Lambda}) = \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{D}^2$ |
| $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ where the λ_i are the 5 - <i>th</i> roots of unity. | $Z^c(\mathbf{\Lambda}) = \sharp_5 \mathbb{S}^3 \times \mathbb{S}^4$ | $Z_0^c(\mathbf{\Lambda}) = \mathbb{T}^2 \times \mathbb{S}^3$ | $Z_+^c(\mathbf{\Lambda}) = (\mathbb{S}^3 \times \mathbb{S}^3 \setminus \mathbb{D}^6) \amalg \tilde{\mathcal{E}}$, where $\tilde{\mathcal{E}}$ is the exterior of the canonical embedded \mathbb{T}^2 in \mathbb{S}^6 . |
| $\mathbf{\Lambda} = (\lambda_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ where the λ_i are the 5 - <i>th</i> roots of unity. | $Z^c(\mathbf{\Lambda}) = \sharp_2 \mathbb{S}^5 \times \mathbb{S}^4 \sharp_3 \mathbb{S} \times \mathbb{S}^6$ | $Z_0^c(\mathbf{\Lambda}) = \sharp_5 \mathbb{S}^3 \times \mathbb{S}^4$ | $Z_+^c(\mathbf{\Lambda}) = \amalg_2 (\mathbb{D}^4 \times \mathbb{S}^4) \amalg \amalg_3 (\mathbb{S}^3 \times \mathbb{D}^5)$ |
| $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$ where λ_i are the 7 - <i>th</i> roots of unity. | $Z^c(\mathbf{\Lambda}) = \sharp_7 \mathbb{S}^5 \times \mathbb{S}^6$ | $Z_0^c(\mathbf{\Lambda}) = \sharp_2 \mathbb{S}^5 \times \mathbb{S}^4 \sharp_3 \mathbb{S}^3 \times \mathbb{S}^6$ | $Z_+^c(\mathbf{\Lambda}) = \amalg_2 (\mathbb{S}^5 \times \mathbb{D}^5) \amalg \left(\amalg_3 \mathbb{D}^4 \times \mathbb{S}^6 \right) \amalg \left((\mathbb{S}^5 \times \mathbb{S}^5) \setminus \mathbb{D}^{10} \right)$ |

Table 1: Open Book Decomposition of some Moment-Angle Manifolds.

4 Complex and algebraic structures on moment-angle manifolds, LV-manifolds and pages.

If we take the quotient of $Z^c(\mathbf{\Lambda})$ by the scalar (or diagonal) action of \mathbb{S}^1 :

$$\mathcal{N}(\mathbf{\Lambda}) = Z^c(\mathbf{\Lambda})/\mathbb{S}^1,$$

we obtain a compact, smooth manifold $\mathcal{N}(\mathbf{\Lambda}) \subset \mathbb{CP}^{n-1}$. These manifolds are called by some authors **LV-M** manifolds.

It is known that various of these objects admit natural complex structures (see, for example, [3], Theorem 12.2): When k is even, both $\mathcal{N}(\mathbf{\Lambda})$ and $Z^c(\mathbf{\Lambda}) \times \mathbb{S}^1$ have natural complex structures. When k is odd, $Z^c(\mathbf{\Lambda})$ itself has a natural complex structure.

Let

$$\pi_{\mathbf{\Lambda}} : Z^c(\mathbf{\Lambda}) \rightarrow \mathcal{N}(\mathbf{\Lambda}),$$

denote the canonical projection.

Consider now the open book decomposition of $Z^c(\mathbf{\Lambda})$ described above, corresponding to the variable z_1 . Let $\mathbf{\Lambda}_0$ be obtained from $\mathbf{\Lambda}$ by removing λ_1 . It is clear that the diagonal \mathbb{S}^1 -action on $Z^c(\mathbf{\Lambda})$ has the property that each orbit intersects the page $Z_+^c(\mathbf{\Lambda})$ in a unique point and every point of this page is intersected transversally by the orbits. This implies that the restriction of the canonical projection $\pi_{\mathbf{\Lambda}}$ to each page is a diffeomorphism onto its image $\mathcal{N}(\mathbf{\Lambda}) - \mathcal{N}(\mathbf{\Lambda}_0)$.

For k even we therefore obtain, by pulling-back the complex structure of $\mathcal{N}(\mathbf{\Lambda}) - \mathcal{N}(\mathbf{\Lambda}_0)$:

Remark 1. *The page of the open book decomposition of $Z^c(\mathbf{\Lambda})$ with binding $Z_0^c(\mathbf{\Lambda}_0)$ admits a natural complex structure which makes it biholomorphic to the complex manifold $\mathcal{N}(\mathbf{\Lambda}) - \mathcal{N}(\mathbf{\Lambda}_0)$.*

We can go a step further by using the results in [21]:

It is shown there that for every $\mathbf{\Lambda}$ the manifold $\mathcal{N}(\mathbf{\Lambda})$ has a holomorphic and locally-free action of $\mathbb{C}^{k/2}$ whose orbits determine a transversally Kähler foliation \mathcal{F} of complex dimension $k/2$. The condition for the leaves of \mathcal{F} to be compact is the following:

An admissible configuration $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n)$ fulfills condition **(K)** if and only if we may choose, for the (real) space of solutions of the system

$$\begin{cases} \sum_{i=1}^n s_i \lambda_i &= 0, \\ \sum_{i=1}^n s_i &= 0, \end{cases}$$

a basis with integer coordinates.

Also it was shown in [21] that every configuration $\mathbf{\Lambda}$ can be made to satisfy condition (\mathbf{K}) by an arbitrarily small perturbation. The main result proved in [21] is the following:

Theorem 4. *Let $\mathbf{\Lambda}$ be an admissible configuration that satisfies condition (\mathbf{K}) . Then*

- (a) *The leaves of the foliation \mathcal{F} of $\mathcal{N}(\mathbf{\Lambda})$ are compact complex tori of complex dimension m .*
- (b) *The quotient space of $\mathcal{N}(\mathbf{\Lambda})$ by \mathcal{F} is a projective toric variety of complex dimension $n - 2m - 1$. We denote it by $T(\mathbf{\Lambda})$.*
- (c) *The toric variety $T(\mathbf{\Lambda})$ comes equipped with an equivariant orbifold structure.*
- (d) *The natural projection $p_{\mathbf{\Lambda}} : \mathcal{N}(\mathbf{\Lambda}) \rightarrow T(\mathbf{\Lambda})$ is a holomorphic principal Seifert bundle, with compact complex tori of complex dimension m as fibers.*
- (e) *The transversely Kählerian form ω of $\mathcal{N}(\mathbf{\Lambda})$ projects onto a Kählerian form $\tilde{\omega}$ of $T(\mathbf{\Lambda})$.*

The bundle $p_{\mathbf{\Lambda}} : \mathcal{N}(\mathbf{\Lambda}) \rightarrow T(\mathbf{\Lambda})$ is called a *generalized Calabi-Eckmann fibration* over $T(\mathbf{\Lambda})$.

If $\mathbf{\Lambda}$ satisfies condition (\mathbf{K}) (and therefore $\mathbf{\Lambda}_0$ also) the manifolds $T(\mathbf{\Lambda})$ and $T(\mathbf{\Lambda}_0)$ are both toric orbifolds (possibly singular) and therefore both are algebraic varieties. As a consequence, $T(\mathbf{\Lambda}) - T(\mathbf{\Lambda}_0)$ is a quasi-projective variety, which (when is nonsingular) is a Kähler manifold and, in particular, symplectic.

Therefore, we have shown:

Theorem 5. *Assume k is even. Then the leaf of the open book structure on $Z^c(\mathbf{\Lambda})$ is naturally a complex manifold. After a small perturbation of $\mathbf{\Lambda}$, this leaf admits a holomorphic (Seifert) fibration over a quasitoric variety, with fibre a compact complex torus.*

5 Different Geometric Structures.

The even dimensional moment-angle manifolds and the \mathbf{LV} -manifolds have the characteristic that, except for a few, well-determined cases, they do not admit

symplectic structures. It was, therefore, surprising to us that the odd dimensional ones can have contact structures. In fact we can conjecture that they are all contact manifolds:

Conjecture 1. *Assume $k = 2m$ is even. Then, for every admissible configuration $\mathbf{\Lambda} \subset \mathbb{R}^{2m}$, the odd dimensional manifold $Z^c(\mathbf{\Lambda})$ is a contact manifold.*

Here is a first example: The next theorem was proved by F. Bourgeois in [4] (it is a corollary of Theorem 10 in [11]).

Theorem 6. *If a closed manifold \mathcal{M} admits a contact structure, then so does $\mathcal{M} \times \mathbb{T}^2$.*

Therefore, For $n > 3$, moment-angle manifolds such as

$$Z^c(\mathbf{\Lambda}) = \mathbb{S}^{2n-5} \times \mathbb{T}^{2m}$$

admit a contact structure.

We will get closer to our conjecture by showing that odd-dimensional moment-angle manifolds admit structures that are weaker versions of the contact structure.

5.1 Almost contact manifolds and quasi-contact manifolds.

Definition 1. A $(2n + 1)$ -dimensional manifold \mathcal{M} is called *almost contact* if its tangent bundle admits a reduction to $\mathbf{U}(n) \times \mathbb{R}$.

It is known that every contact manifold is an almost contact manifold (see [14]).

On the other hand, D. Martínez, V. Muñoz and F. Presas in [18] defined a quasi-contact manifold as follows.

Definition 2. A $(2n + 1)$ -dimensional manifold \mathcal{M} is called *quasi-contact* if it admits a closed 2-form ω such that ω^n is a non-zero $2n$ -form all over the manifold.

They proved that given an almost contact manifold \mathcal{M} and given $\gamma \in H^2(\mathcal{M}, \mathbb{R})$, there exist a quasi-contact structure ω in \mathcal{M} such that $[\omega] = \gamma$. They also conjectured that every almost contact manifold is in fact a contact manifold.

Now, we have:

Theorem 7. *If k is even, $Z^c(\mathbf{\Lambda})$ is an almost contact manifold and also a quasi-contact manifold.*

Proof. Consider the fibration $\pi_{\mathbf{\Lambda}} : Z^c(\mathbf{\Lambda}) \rightarrow \mathcal{N}(\mathbf{\Lambda})$ with fibre the circle, given by taking the quotient by the diagonal action. Since $N(\mathbf{\Lambda})$ is a complex manifold we have that the foliation defined by the diagonal circle action is transversally

holomorphic. Therefore, $Z^\mathbb{C}(\mathbf{A})$ has an atlas modeled on $\mathbb{C}^{n-2} \times \mathbb{R}$ with changes of coordinates of the charts of the form

$$((z_1, \dots, z_{n-2}), t) \mapsto (h(z_1, \dots, z_{n-2}, t), g(z_1, \dots, z_{n-2}, t)),$$

where $h : U \rightarrow \mathbb{C}^{n-2}$ and $g : U \rightarrow \mathbb{R}$ where U is an open set in $\mathbb{C}^{n-2} \times \mathbb{R}$ and, for each fixed t the function $(z_1, \dots, z_{n-2}) \mapsto h(z_1, \dots, z_{n-2}, t)$ is a biholomorphism onto an open set of $\mathbb{C}^{n-2} \times \{t\}$. This means that the differential, in the given coordinates, is represented by a matrix of the form

$$\left[\begin{array}{ccc|c} & & & \\ & A & & * \\ \hline 0 & \dots & 0 & r \end{array} \right],$$

where $*$ denotes a column $(n-2)$ -real vector and $A \in \mathbf{GL}(n-2, \mathbb{C})$. The set of matrices of the above type form a subgroup of $\mathbf{GL}(2n-3, \mathbb{R})$. By Gram-Schmidt this group retracts onto $\mathbf{U}(n-2) \times \mathbb{R}$.

This shows that $Z^\mathbb{C}(\mathbf{A})$ is an almost contact manifold and therefore also a quasi-contact manifold by the result in [18]. \blacksquare

Remark 2.

1. The conjecture in [18] would imply our conjecture that every odd dimensional moment-angle manifold admits a contact structure. We are working on a direct proof of this. It would give support to the conjecture in [18]. In section 6 we will construct contact structures in some manifolds related to moment-angle manifolds.
2. The construction in [18] is based on an open book decomposition of the manifold. We have still not related this open book structure with the one we have in the case of a moment-angle manifold.

5.2 PS-Overtwisted Manifolds.

The definition of *overtwistedness* in higher dimensions was given by K. Niederkrüger based in the existence of a *plastikstufe* (see [22]).

Definition 3. Let (\mathcal{M}, α) be a cooriented $(2n+1)$ -dimensional contact manifold, and let S be a closed $(n-2)$ -dimensional manifold. A *plastikstufe* $\mathbf{PS}(S)$ with singular set S in \mathcal{M} is an embedding of the n -dimensional manifold

$$\iota : \mathbb{D}^2 \times \mathbb{S}^1 \rightarrow \mathcal{M},$$

that carries a (singular) Legendrian foliation given by the 1-form $\beta := \iota^* \alpha$ satisfying:

- a) The boundary $\partial \mathbf{PS}(S)$ of the plastikstufe is the only closed leaf.

- b) There is an elliptic singular set at $0 \times S$.
- c) The rest of the plastikstufe is foliated by an \mathbb{S}^1 -family of stripes, each one diffeomorphic to $(0, 1) \times S$, which are spanned between the singular set on one end and approach $\partial \mathbf{PS}(S)$ on the other side asymptotically.

A contact manifold is called **PS-overtwisted** if it admits the embedding of a plastikstufe.

Observation 1. In dimension three, the definition of **PS-overtwisted** is identical to the standard definition of overtwistedness.



Figure 1: An overtwisted disk.

Following the ideas of F. Presas in the article [24], K. Niederküger and O. Van Koert proved the next interesting results (see [23]):

Let \mathbb{S}^{2n-1} be the unit sphere in \mathbb{C}^n with coordinates $Z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and let f be the polynomial

$$f : \mathbb{C}^n \rightarrow \mathbb{C} \quad \text{defined by} \quad (z_1, \dots, z_n) \mapsto z_1^2 + \dots + z_n^2.$$

The 1-form

$$\alpha_- := i \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j) - i (f d\bar{f} - \bar{f} df)$$

defines a contact structure on \mathbb{S}^{2n-1} (see [23, Proposition 7]).

Theorem 8. [23, Corollary 4] *Every sphere \mathbb{S}^{2n+1} with $n \geq 1$ supports a PS-overtwisted contact structure. More precisely, on \mathbb{S}^{2n+1} , with $n \geq 1$, exists a contact structure which admits the embedding of a plastikstufe $PS(\mathbb{T}^{n-1})$ (with $\mathbb{T}^0 := \{p\}$ and $\mathbb{T}^1 := \mathbb{S}^1$).*

As a consequence of Theorem 6, we have the next corollary:

Corollary 1. *For $n > 3$, moment-angle manifolds such as $Z^c(\mathbf{\Lambda}) = \mathbb{S}^{2n-5} \times \mathbb{T}^{2m}$ are **PS-overtwisted** contact manifolds.*

6 Higher Dimensional Contact Manifolds.

Theorem 2 of [16] was extended in [15] to the case where the manifold is given by two quadratic forms which are not necessarily simultaneously diagonalizable. This includes the manifolds we construct now:

Let $n > 3$ and $s \geq 1$ be two integers and $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n)$ be an admissible configuration with $\lambda_1 \in \mathbb{C}$. Now consider the manifold $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ define by the following equations:

$$F_s(X) := \sum_{r=1}^s w_r^2 + \sum_{j=1}^n \lambda_j |z_j|^2 = 0, \quad (1)$$

$$\rho_s(X) := \sum_{r=1}^s |w_r|^2 + \sum_{j=1}^n |z_j|^2 = 1, \quad (2)$$

where $X = (w_1, \dots, w_s, z_1, \dots, z_n)$. $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ has real dimension $2n + 2s - 3 > 5$.

The topology of the manifolds $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ is related to the topology of the associated moment-angle manifolds $Z^{\mathbb{C}}(\mathbf{\Lambda})$ as shown by the following theorem.

Theorem 9 ([15]). *The manifold $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ is diffeomorphic to:*

$$\bigg\#_{j=1}^{2\ell+1} (\mathbb{S}^{2d_j+s-1} \times \mathbb{S}^{2n-2d_j+s-2}),$$

where $d_j = n_j + \dots + n_{j+\ell-1}$.

We will show that for every set of $n + s$ positive numbers $a_1, \dots, a_s, b_1, \dots, b_n$ the 1-form

$$\alpha := i \left[\sum_{r=1}^s [a_r (w_r d\bar{w}_r - \bar{w}_r dw_r)] + \sum_{j=1}^n [b_j (z_j \bar{z}_j - \bar{z}_j dz_j)] \right]$$

can be deformed into a contact form on the manifold $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$, by an arbitrarily small \mathbb{C}^∞ perturbation.

6.1 Case $s = 1$.

Lemma 1. *For all $X = (w_1, z_1, \dots, z_n) \in \mathbf{Z}_{(\mathbf{\Lambda}, n, 1)}^{\mathbb{C}}$ the 1-form α restricted to the tangent space $\mathbf{T}_X(\mathbf{Z}_{(\mathbf{\Lambda}, n, 1)}^{\mathbb{C}})$ is a nontrivial form.*

We will only consider the case where $a_1 = 1$ and $b_1 = 1$, $j = 1, \dots, n$ since the other cases are completely analogous.

Proof. For $X = (w_1, z_1, \dots, z_n) \in \mathbf{Z}_{(\Lambda, n, 1)}^c$, the linear function

$$\alpha_X : \mathbf{T}_X \left(\mathbf{Z}_{(\Lambda, n, 1)}^c \right) \rightarrow \mathbb{R}$$

is trivial if and only if there exist $T \in \mathbb{C}$ and $\mu \in \mathbb{R}$ such that:

$$\begin{aligned} & i \left[(w_1 d\bar{w}_1 - \bar{w}_1 dw_1) + \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j) \right] \\ &= T \left[2w_1 dw_1 + \sum_{j=1}^n [\lambda_j (\bar{z}_j dz_j + z_j d\bar{z}_j)] \right] \\ &+ \bar{T} \left[2\bar{w}_1 d\bar{w}_1 + \sum_{j=1}^n [\bar{\lambda}_j (z_j d\bar{z}_j + \bar{z}_j dz_j)] \right] \\ &+ \mu \left[(w_1 d\bar{w}_1 + \bar{w}_1 dw_1) + \sum_{j=1}^n (z_j d\bar{z}_j + \bar{z}_j dz_j) \right]. \end{aligned} \quad (3)$$

Comparing coefficients in equation (3) we conclude that the unique point satisfying the last equations is the origin $(0, 0, \dots, 0)$. However, the origin is not in $\mathbf{Z}_{(\Lambda, n, 1)}^c$. We conclude that α is nontrivial. \blacksquare

We will denote by $\xi_\alpha(X)$ the kernel of α at the point $X \in \mathbf{Z}_{(\Lambda, n, 1)}^c$.

The only vectors $v \in \mathbf{T} \left(\mathbf{Z}_{(\Lambda, n, 1)}^c \right)$ where $\iota_v d\alpha = 0$ are of the form:

$$v(T, \mu) := -i \left(2\bar{T}\bar{w}_1 + \mu w_1, (2\Re(T\lambda_1) + \mu)z_1, \dots, (2\Re(T\lambda_n) + \mu)z_n \right),$$

where $T \in \mathbb{C}$ and $t \in \mathbb{R}$.

If $v(T, \mu)$ is tangent to the sphere \mathbb{S}^{2n-1} and $w_0 \neq 0$ then it follows that T must be of the form

$$T = t\bar{w}_1^2$$

with $t \in \mathbb{R}$.

The condition $dF_1(v(T, \mu)) = 0$, when $w_1 \neq 0$ and $v(T, \mu) \in \mathbb{S}^{2n+1}$ implies:

$$2tw_1^2|w_1|^2 + \mu w_1^2 = 0,$$

which implies since $w_1 \neq 0$ that

$$\mu = -2t|w_1|^2$$

if $w_1 = x + iy$ we have:

$$\mu = -2t(x^2 + y^2).$$

Therefore, if $\mu = 0$ implies $t = 0$ since, by hypothesis, what is inside the parentheses is positive.

Vectors of the form

$$v(T, \mu) = -i \left(2\bar{T}\bar{w}_1 + \mu w_1, (2\Re(T\lambda_1) + \mu)z_1, \dots, (2\Re(T\lambda_n) + \mu)z_n \right),$$

which are in $\ker \alpha$, imply that $\mu = 0$. Hence, if $w_1 \neq 0$ we must have that $t = 0$ also.

For $w_1 = 0$ the vectors of the form $v(T, \mu)$ with $T \in \mathbb{C}$ and $\mu \in \mathbb{R}$ are in $\ker(d\alpha)$. Hence $\ker(d\alpha)$ has real dimension three.

On the other hand, on the set of points with $w_1 = 0$, the vectors of the form:

$$-i(0, 2\Re(T\lambda_1)z_1, \dots, 2\Re(T\lambda_n)z_n), \quad T \in \mathbb{C}$$

are all in $\ker(\alpha) \cap \ker(d\alpha)$.

Hence $\ker(\alpha) \cap \ker(d\alpha)$ is a two dimensional vector space. We will denote this 2-dimensional vector space at the point $X = (0, z_1, \dots, z_n)$ by $\Pi(X)$.

Therefore, in the set of points of $\mathbf{Z}_{(\mathbf{\Lambda}, n, 1)}^{\mathbb{C}}$ such that $w_1 \neq 0$, the form α is a contact form.

For a generic set of admissible configurations $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n)$ the intersection of $\mathbf{Z}_{(\mathbf{\Lambda}, n, 1)}^{\mathbb{C}}$ with the complex hyperplane with equation $w_1 = 0$ is a real codimension-two submanifold W of $\mathbf{Z}_{(\mathbf{\Lambda}, n, 1)}^{\mathbb{C}}$ (i.e. a submanifold of \mathbb{C}^{n+1} of dimension $2n-3$) and $\Pi(X) \subset \mathbf{T}_X(W)$.

Remark 3. W is essentially the moment-angle manifold $Z^{\mathbb{C}}(\mathbf{\Lambda}') \subset \mathbb{C}^n$ where $\mathbf{\Lambda}' = (\lambda_1, \dots, \lambda_n)$.

6.2 Case $s > 1$.

Lemma 2. For all $X = (w_1, \dots, w_s, z_1, \dots, z_n) \in \mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ the 1-form α restricted to the tangent space $\mathbf{T}_X(\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}})$ is a non trivial form.

As in the case $s = 1$, we will consider $a_r = b_j = 1$ for all $r \in (1, \dots, s)$ and $j \in (1, \dots, n)$ since the other cases are completely analogous.

Proof. For $X = (w_1, \dots, w_s, z_1, \dots, z_n) \in \mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$, the linear function

$$\alpha : \mathbf{T}_X(\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}) \rightarrow \mathbb{R}$$

is trivial if and only if there exist $T \in \mathbb{C}$ and $\mu \in \mathbb{R}$ such that

$$\begin{aligned}
& i \left[\sum_{r=1}^s (w_r d\bar{w}_r - \bar{w}_r dw_r) + \sum_{j=1}^n (z_j d\bar{z}_j - \bar{z}_j dz_j) \right] \\
&= T \left[\sum_{r=1}^s 2w_r dw_r + \sum_{j=1}^n [\lambda_j (\bar{z}_j dz_j + z_j \bar{z}_j)] \right] \\
&+ \bar{T} \left[\sum_{r=1}^s 2\bar{w}_r d\bar{w}_r + \sum_{j=1}^n [\lambda_j (z_j d\bar{z}_j - \bar{z}_j dz_j)] \right] \\
&+ \mu \left[\sum_{r=1}^s (w_r d\bar{w}_r + \bar{w}_r dw_r) + \sum_{j=1}^n (z_j d\bar{z}_j + \bar{z}_j dz_j) \right]. \tag{4}
\end{aligned}$$

Comparing coefficients we have that the points satisfying the last equations are of the form $(w_1, \dots, w_s, 0, \dots, 0)$ where at least two $w_r, w_{r'}$ not zero, $r \neq r', r, r' \in \{1, \dots, s\}$. In this cases, the equations $F_s(X) = 0$ and $\rho_s(X) = 1$ defines a Brieskorn manifold and α is a contact form on that manifolds [17], in particular is non trivial on it.

We conclude that α is non trivial on $\mathbf{Z}_{(\mathbf{A}, n, s)}^c$. ■

We will denote by $\xi_\alpha(X)$ the kernel of α at the point $X \in \mathbf{Z}_{(\mathbf{A}, n, s)}^c$.

The unique vectors $v \in \mathbf{T}(\mathbf{Z}_{(\mathbf{A}, n, s)}^c)$ where $\iota_v d\alpha = 0$ are of the form:

$$v(T, \mu) :=$$

$$-i(2\bar{T}\bar{w}_1 + \mu w_1, \dots, 2\bar{T}\bar{w}_s + \mu w_s, (2\Re(T\lambda_1) + \mu)z_1, \dots, (2\Re(T\lambda_n) + \mu)z_n),$$

where $T \in \mathbb{C}$ and $\mu \in \mathbb{R}$.

If $v(T, \mu)$ is tangent to the sphere $\mathbb{S}^{2n+2s-1}$ and $w_r \neq 0$ for all $r \in \{1, \dots, s\}$ then it follows that T must be of the form

$$T = t \sum_{r=1}^s \bar{w}_r^2, \quad t \in \mathbb{R}.$$

The condition $dF_s(v) = 0$ when $w_r \neq 0$ for all r and $v(T, \mu) \in \mathbb{S}^{2n+2s-1}$ implies:

$$2t \sum_{r=1}^s \left(\sum_{r=1}^s w_r^2 \right) |w_r|^2 + \mu \sum_{r=1}^s w_r^2 = 0,$$

which implies, since $w_r \neq 0$, that

$$\mu = -2t \sum_{r=1}^s |w_r|^2.$$

if $w_r = x_r + iy_r$ we have:

$$\mu = -2t \sum_{r=1}^s (x_r^2 + y_r^2).$$

Therefore, if $\mu = 0$ implies $t = 0$ since, by hypothesis, $\sum_{r=1}^s (x_r^2 + y_r^2)$ is positive.

Vectors of the form $v(T, \mu)$ which are in $\ker(\alpha)$ imply that $\mu = 0$. Hence, if $w_r \neq 0$ we must have that $t = 0$ also.

When $w_r = 0$ for all $r \in \{1, \dots, s\}$, the vectors of the form $v(T, \mu)$ with $T \in \mathbb{C}$ and $\mu \in \mathbb{R}$ are in $\ker(d\alpha)$. Hence $\ker(d\alpha)$ has real dimension three.

On the other hand, on the set of points with $w_r = 0$ for all $r \in \{1, \dots, s\}$, the vectors of the form

$$-i(0, \dots, 0, 2\Re(T\lambda_1)z_1, \dots, 2\Re(T\lambda_n)z_n), \quad T \in \mathbb{C}$$

are all in $\ker(\alpha) \cap \ker(d\alpha)$.

Hence $\ker(\alpha) \cap \ker(d\alpha)$ is a two dimensional space. We will denote this 2-dimensional vector space at the point $X = (0, \dots, 0, z_1, \dots, z_n)$ by $\Pi(X)$.

Therefore, in the set of points of $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ such that $w_r \neq 0$ for all $r \in \{1, \dots, s\}$, the form α is a contact form.

For a generic set of admissible configurations $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n)$ the intersection of $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$ with the complex hyperplanes $\{w_1 = 0\}, \dots, \{w_s = 0\}$ is a submanifold W of real codimension $2s$ of $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}}$.

6.3 Conductive Confoliations.

We introduce some definitions following the ideas of S. J. Altschuler and L. F. Wu in [1]:

Definition 4. The space of *conductive confoliations*, $Con(\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}})$, is defined to be the subset of $\alpha \in \mathbf{\Lambda}^1(\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^{\mathbb{C}})$ such that

- α is a positive confoliation: $\ast(\alpha \wedge (d\alpha)^{n+s-2}) \geq 0$, where \ast denotes de Hodge operator;

- every point $p \in \mathbf{Z}_{(\mathbf{A},n,s)}^c$ is accessible from a contact point $q \in \mathbf{Z}_{(\mathbf{A},n,s)}^c$ of α : there is a smooth path $\gamma : [0, 1] \rightarrow \mathbf{Z}_{(\mathbf{A},n,s)}^c$ from p to q with $\gamma'(s)$ in the orthogonal complement of $\ker (*(\alpha \wedge (d\alpha)^{n+s-3}))$ for all s .

Theorem 10 (Theorem 2.8, [1]). *If $\alpha \in \text{Con}(\mathbf{Z}_{(\mathbf{A},n,s)}^c)$ then α is C^∞ close to a contact form.*

From the above and the fact that W does not separate $\mathbf{Z}_{(\mathbf{A},n,s)}^c$, since it is of codimension two, it follows:

Proposition 1. *Let $*$ denote the Hodge operator for a given Riemannian metric on $\mathbf{Z}_{(\mathbf{A},n,s)}^c$, then for the appropriate orientation of $\mathbf{Z}_{(\mathbf{A},n,s)}^c$ one has that*

$$*(\alpha \wedge (d\alpha)^{n+s-2})(X) > 0,$$

for $X \notin W$ and if $X \in W$

$$*(\alpha \wedge (d\alpha)^{n+s-2})(X) = 0.$$

Therefore α is a positive confoliation on $\mathbf{Z}_{(\mathbf{A},n,s)}^c$.

We must show that every point is accessible from a contact point of α .

Lemma 3. *Let $X \in W$. Then, there exists a smooth parametrized curve $\gamma : [0, 1] \rightarrow \mathbf{Z}_{(\mathbf{A},n,s)}^c$ such that $\gamma(0) = X$, $\gamma(s) \notin W$ if $s \in (0, 1]$ and $\gamma'(s)$ is a nonzero vector such that $\gamma'(s) \in \xi_\alpha(\gamma(s))$.*

Proof. Let us fix a Riemannian metric g . For $P \in W$ let $\mathbf{N}(P)$ denote the 2-dimensional subspace of $\mathbf{T}(\mathbf{Z}_{(\mathbf{A},n,s)}^c)$ which is orthogonal to $\mathbf{T}_P W$ at P .

Let us first show that there exists an open neighborhood $\mathcal{U} \subset W$ of $X \in W$ and a smooth and non-vanishing vector field $\mathcal{X} : \mathcal{U} \rightarrow \mathbf{T}_X(\mathbf{Z}_{(\mathbf{A},n,s)}^c)$ defined on \mathcal{U} such that

1. $\mathcal{X}(X) \in \xi_\alpha(X) \cap \mathbf{N}(X)$,
2. $\mathcal{X}(P) \in \xi_\alpha(P)$ for all $P \in \mathcal{U}$,
3. $\mathcal{X}(P) \notin \mathbf{T}_P(W)$ for all $P \in \mathcal{U}$.

Indeed, Let $\mathcal{L}(X) = \mathbf{N}(X) \cap \xi_\alpha(X)$. Then $\mathcal{L}(X)$ has dimension two if $\mathbf{N}(X) \subset \xi_\alpha(X)$ or $\mathcal{L}(X)$ has dimension one if $\mathbf{N}(X)$ is transverse to $\xi_\alpha(X)$.

Let $v_X \in \mathcal{L}(X)$ be a non zero vector. Extend this vector anchored at X to a smooth vector field $\tilde{\mathcal{X}} : \mathcal{K} \rightarrow \mathbf{T}(\mathbf{Z}_{(\mathbf{A},n,s)}^c)$ defined in a neighborhood $X \in \mathcal{K} \subset W$ of $X \in W$. Let $\pi_P : \mathbf{T}_P(\mathbf{Z}_{(\mathbf{A},n,s)}^c) \rightarrow \xi_\alpha(P)$ be the orthogonal

projection. Consider the vector field defined on \mathcal{K} by $\mathcal{X}_1(P) = \pi_P(\tilde{\mathcal{X}}(P))$. Then \mathcal{X}_1 is a smooth vector field and by continuity \mathcal{X}_1 satisfies all the required properties in a possible smaller neighborhood \mathcal{U} .

To finish the proof of the lemma we have that by standard extension theorems (partition of unity) there exists an extension of \mathcal{X}_1 to a nonsingular vector field $\mathcal{X}_2 : \mathcal{V} \rightarrow \mathbf{T}_X \left(\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c \right)$ defined on an open neighborhood $X \in \mathcal{V} \subset \mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c$ of X . The vector field defined by $\mathcal{X}(P) = \pi_P(\mathcal{X}_2(P))$ has the property that $\mathcal{X}(P) \in \xi_\alpha(P)$.

By multiplying the vector field \mathcal{X} by a positive constant $c > 0$, if necessary, we can assume that all the solutions of the differential equation defined by the vector field $c\mathcal{X}$ on \mathcal{V} are defined in the interval $(-2, 2)$.

If $\gamma : [0, 1] \rightarrow \mathcal{U}$ is the solution of the differential equation determined by $c\mathcal{X}$ and satisfying the initial condition $\gamma(0) = X$ then this parametrized curve satisfies the requirements of the theorem 10 if c is sufficiently small. ■

The proposition implies that every point of W can be joined, by a Legendrian path of finite length, to a point where the form $\ast (\alpha \wedge (d\alpha)^{n+s-2}) (X)$ is positive.

Since α is a contact form on $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c - W$ we have that any two points of $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c - W$ can be joined by a Legendrian curve. Therefore every point of $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c$ is *accessible* and α defines a *conductive confoliation* in the sense of J. S. Altschuler and L. F. Wu ([1]).

These forms are also called *transitive confoliations* by Y. Eliashberg and W. P. Thurston [9], since we can connect any point of the manifold to a point where the form α is contact by a Legendrian path of finite length.

Therefore applying Theorem 10 we can deform α to a contact form α' . Furthermore α' can be chosen arbitrarily close to α in the C^∞ topology. We have the theorem:

Theorem 11. *For a generic set of admissible configurations $\mathbf{\Lambda} = (\lambda_1, \dots, \lambda_n)$, the manifold $\mathbf{Z}_{(\mathbf{\Lambda}, n, s)}^c$ is a contact manifold.*

In other words:

Theorem 12. *Let $n > 3$, $s \in \{1, \dots, n\}$ and let $\mathbf{\Lambda}$ be an admissible configuration. The manifolds*

$$\biguplus_{j=1}^{2\ell+1} (\mathbb{S}^{2d_j+s-1} \times \mathbb{S}^{2n-2d_j+s-2}),$$

where $d_j = n_j + \dots + n_{j+\ell-1}$ admit contact structures.

Remark 4. It was shown by C. Meckert [19] that the connected sum of contact manifolds of the same dimension is a contact manifold. It was pointed to us by Dishant Pancholi that this implies the manifolds $Z_{(\mathbf{A},n,s)}^c$ have a contact structure.

Indeed, the manifolds $Z_{(\mathbf{A},n,s)}^c$ are connected sums of products of the form $\mathbb{S}^n \times \mathbb{S}^m$ with n even and m odd, and $n, m > 2$. Without loss of generality, we suppose that $m > n$ (the other case is analogous) then \mathbb{S}^m is an open book with binding \mathbb{S}^{m-2} and page \mathbb{R}^{m-1} . Hence $\mathbb{S}^n \times \mathbb{S}^m$ is an open book with binding $\mathbb{S}^{m-2} \times \mathbb{S}^n$ and page $\mathbb{R}^{m-1} \times \mathbb{S}^n$. The page $\mathbb{R}^{m-1} \times \mathbb{S}^n$ is parallelizable since it embeds as an open subset of \mathbb{R}^{m+n-1} , therefore, since $m+n-1$ is even it has an almost complex structure. Furthermore, by hypothesis, $2n \leq n+m$ hence by a theorem of Y. Eliashberg [8] the page is Stein and is the interior of an compact manifold with contact boundary $\mathbb{S}^{m-2} \times \mathbb{S}^n$. Hence by a theorem of E. Giroux [11] $\mathbb{S}^n \times \mathbb{S}^m$ is a contact manifold. However our construction is in some sense explicit since it is the instantaneous diffusion through the heat flow of an explicit 1-form which is a positive confoliation.

Remark 5. Another interesting fact is that the manifolds $Z_{(\mathbf{A},n,s)}^c$ also have an open book decomposition. However for these open book decompositions there does not exist a contact form which is supported in the open book decomposition like in Giroux's theorem because the pages are not Weinstein manifolds (i.e manifolds of dimension $2n$ with a Morse function with indices of critical points lesser or equal to n).

7 Topology of Intersections of quadrics.

In this last section we recall some old results about the topology of intersections of quadrics $Z(\mathbf{A})$ and the ideas behind their proofs. We use these to proof a new result about the topology of the manifolds with boundary $Z_+(\mathbf{A})$ from which Theorem 3 of section 2 follows.

7.1 Homology of Intersections of quadrics.

We recall here the results of [16], whose proofs are equally valid for any intersection of quadrics and not only for $k = 2$:

Let $Z = Z(\mathbf{A}) \subset \mathbb{R}^n$ as before, \mathcal{P} its associated polytope and F_1, \dots, F_n its facets obtained by intersecting \mathcal{P} with the coordinate hyperplanes $x_i = 0$ (some of which might be empty).

Let g_i be the reflection on the i -th coordinate hyperplane and for $J \subset \{1, \dots, n\}$ let g_J be the composition of the g_i with $i \in J$.

Let also F_J be the face obtained by intersecting the facets F_i for $i \in J$.

The polytope \mathcal{P} , all its faces, and all their combined reflections on the coordinate hyperplanes form a cell decomposition of Z . Then the elements $g_J(F_L)$ are generators of the chain groups $C_*(Z)$, where to avoid repetitions one has to ask $J \cap L = \emptyset$ (since for any i the intersection g_i acts trivially on F_i).

A more useful basis is given as follows: let $h_i = 1 - g_i$ and h_J be the product of the h_i with $i \in J$. The elements $h_J(F_L)$ with $J \cap L = \emptyset$ are also a basis, with the advantage that $h_J C_*(Z)$ is a chain subcomplex for every J and, since h_i annihilates F_i and all its subfaces, it can be identified with the chain complex $C_*(\mathcal{P}, \mathcal{P}_J)$, where \mathcal{P}_J is the union of all the facets F_i with $i \in J$. It follows that

$$H_*(Z) \approx \oplus_J H_*(\mathcal{P}, \mathcal{P}_J).$$

For the manifold Z_+ we have the same elements, but we cannot reflect them in the subspace $x_1 = 0$. This means we miss the classes $h_J(F_L)$ where $1 \in J$ and we get³

$$H_*(Z_+) \approx \oplus_{1 \notin J} H_*(\mathcal{P}, \mathcal{P}_J).$$

These splittings are consistent with the ones derived from the homotopy splitting of ΣZ described in [2]. Even if it is not clear that they are the *same* splitting, having two such with different geometric interpretations is most valuable.

7.2 Topology of Z when $k = 2$.

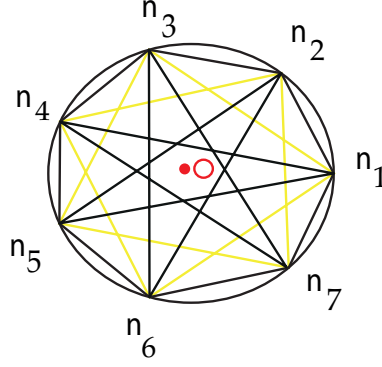
Let us recall the results of [16] for $k = 2$. In this case, every intersection of quadrics is diffeomorphic to one of the following particular forms:

Take $n = n_1 + \dots + n_{2\ell+1}$ a partition of n , where we will not distinguish between a partition and any of its permutations preserving the cyclic order and we will think of the index i in n_i as an interger mod $2\ell + 1$. Corresponding to it we have the configuration \mathbf{A} consisting of the $(2\ell + 1)$ -th roots of unity, the i -th one taken with multiplicity n_i . In other words, we have $\{1, \dots, n\} = J_1 \cup \dots \cup J_{2\ell+1}$ in increasing order where the size of J_i is n_i and for every $j \in J_i$ one has λ_j is the i -th of the $(2\ell + 1)$ -th roots of unity.

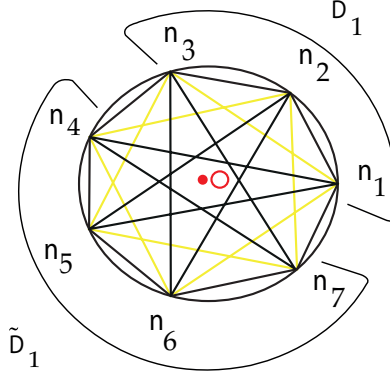
Any configuration can be deformed into one of these by concentrating in one point all the coefficients that is possible without breaking the weak hyperbolicity condition. We think of the J_i as classes of points that can be joined this way.

The pairs $(\mathcal{P}, \mathcal{P}_J)$ with non-trivial homology are those where J consists of ℓ or $\ell + 1$ consecutive classes, that is, those where J is either one of the $D_i = J_i \cup \dots \cup J_{i+\ell-1}$ or one of their complements \tilde{D}_i in $\{1, \dots, n\}$. In those cases

³The retraction $Z \rightarrow Z_+$ induces an epimorphism in homology and fundamental group.



there is just one dimension where the homology is non-trivial and it is infinite cyclic.



In the case of D_i that homology group is in dimension $d_i - 1$ where $d_i = n_i + \dots + n_{i+\ell-1}$ is the length of D_i . To specify a generator consider the cell F_{L_i} where $L_i = \tilde{D}_i \setminus (\{j_{i-1}\} \cup \{j_{i+\ell}\})$ and $j_{i-1} \in J_{i-1}$, $j_{i+\ell} \in J_{i+\ell}$ are any two indices in the extreme classes of \tilde{D}_i (in other words, those contiguous to D_i).

F_{L_i} is non empty of dimension $d_i - 1$. It is not in \mathcal{P}_{D_i} , but its boundary is. Therefore it represents a homology class in $H_{d_i-1}(\mathcal{P}, \mathcal{P}_{D_i})$, which is actually a generator, and defines a generator $h_{D_i} F_{L_i}$ of $H_{d_i-1}(Z)$. Since F_{L_i} has exactly d_i facets it is a $d_i - 1$ -simplex so when reflected in all the coordinate subspaces containing those facets we obtain a sphere, which clearly represents $h_{D_i} F_{L_i} \in H_{d_i-1}(Z)$.

The class corresponding to \tilde{D}_i is in dimension $n - d_i - 2$ and is the Poincaré dual of the the one corresponding to D_i . One gets again easily a representative

which in this case is not a sphere, but which can be turned into one with a good choice and a surgery if $\ell > 1$. For our purpose we do not need to say more about this case.

The net result is that, if $\ell > 1$, all the homology below the top dimension can be represented by embedded spheres⁴ with trivial normal bundle which can be made disjoint inside Z'_+ and (since the inclusion $Z \subset Z'_+$ induces an epimorphism in homology due to the homotopy equivalence $Z_+ \subset Z'_+$) we represent all the classes in $H_*(Z'_+)$ by spheres, the h -cobordism theorem shows that this manifold is a connected sum along the boundary of manifolds of the form $\mathbb{S}^p \times \mathbb{D}^{n-3-p}$. Then its boundary Z is a connected sum of spheres products. Knowing its homology we can tell the dimensions of those spheres:

If $\ell > 1$ and Z is simply connected of dimension at least 5, then⁵:

$$Z = \sum_{j=1}^{2\ell+1} (\mathbb{S}^{d_i-1} \times \mathbb{S}^{n-d_i-2}).$$

When $\ell = 1$ a simple computation shows that

$$Z = \mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1} \times \mathbb{S}^{2n_3-1}.$$

7.3 Topology of Z_+ when $k = 2$.

The topology of Z'_+ is implicit in the above proof, and since any Z with $n_1 > 1$ is such a Z' so we have:

If Z_0 is simply connected of dimension at least 5, and $\ell > 1$, $n_1 > 1$ then:

$$Z_+ = \prod_{i=2}^{\ell+2} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \prod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2}).$$

The case $n_1 = 1$ has to be considered separately. The difference in the topology of Z_+ between case $n > 1$ and $n = 1$ can be seen as follows:

As mentioned before, in the first case the map $Z_0 \rightarrow Z_+$ induces an epimorphism in homology.

This is not the case for $n_1 = 1$. For example, take the case $5 = 1 + 1 + 1 + 1 + 1$. Here Z_0 consists of four copies of \mathbb{S}^1 while Z_+ is a torus minus four disks⁶ Or, equivalently, a sphere minus four disks (where all the homology comes from the

⁴This also follows from [13].

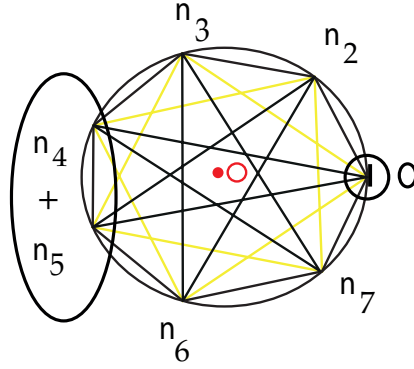
⁵The result has recently been proved in [15] without the dimension and connectivity hypotheses

⁶In this case the polytope is a pentagon and the Euler characteristics of Z and Z_0 follow easily by looking at its faces.

boundary) with a handle attached that carries the homology not coming from the boundary.

The main fact is that Z_0 is given by $2\ell - 1$ classes, and has only $4\ell - 2$ homology generators below the top dimension, only half of which survive in Z_+ . While the latter manifold has $2\ell + 1$ homology generators.

To be more precise, the removal of the element $1 \in I_1$ allows the opposite classes $I_{\ell+1}$ and $I_{\ell+2}$ to be joined into one without breaking the weak hyperbolicity condition.



Therefore Z_0 has fewer such classes and $D_2 = I_2 \cup \dots \cup I_{\ell+1}$, which gives a generator of $H_*(Z_+)$, does not give anything in $H_*(Z_0)$ because there *it is not a union of classes* (it lacks the elements of $I_{\ell+2}$ to be so).

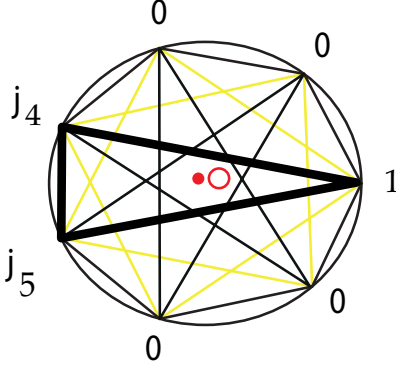
The two classes in $H_*(Z_+)$ missing in $H_*(Z_0)$ are thus those corresponding to $J = D_2$ and $J = D_{\ell+2}$, all the others contain both $I_{\ell+1}$ and $I_{\ell+2}$ and thus live in $H_*(Z_0)$.

As shown above, these two classes are represented by embedded spheres in Z_+ with trivial normal bundle built from the cells F_{L_2} and $F_{L_{\ell+2}}$ by reflection, where

$$L_2 = \tilde{D}_2 \setminus (\{1\} \cup \{j_{\ell+2}\}), 1 \in J_1, j_{\ell+2} \in J_{\ell+2}.$$

$$L_{\ell+2} = \tilde{D}_{\ell+2} \setminus (\{j_{\ell+1} \cup \{1\}\}), j_{\ell+1} \in J_{\ell+1}, 1 \in J_{2\ell+2} = J_1.$$

$F_{L_2} \cap F_{L_{\ell+2}}$ is a single vertex v , all coordinates except $x_1, x_{j_{\ell+1}}, x_{j_{\ell+2}}$ being 0.



The corresponding spheres are obtained by reflecting in the hyperplanes corresponding to elements in D_2 and $D_{\ell+2}$, respectively. Since these sets are disjoint, the spheres also intersect in a single point.

Now, a neighborhood of the vertex v in \mathcal{P} looks like the first orthant of \mathbb{R}^{n-3} where the faces F_{L_2} and $F_{L_{\ell+2}}$ correspond to complementary subspaces. When reflected in all the coordinates hyperplanes of \mathbb{R}^{n-3} , one obtains a neighborhood of v in Z_+ where those subspaces produce neighborhoods of the two spheres. Therefore the spheres intersect transversely in that point. (This shows again that those two classes do not come from the boundary Z_0 : any homology class coming from the boundary can be separated from any other homology class in Z_+ and so has trivial homology intersection with it).

A regular neighborhood of the union of those spheres is diffeomorphic to their product minus a disk, specifically to

$$\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_{\ell+2}-1} \setminus \mathbb{D}^{n-3}.$$

If $\ell > 2$ the rest of the classes coming from Z_0 can be represented again by disjoint products $\mathbb{S}^p \times \mathbb{D}^{n-p-3}$ so finally the h -cobordism theorem gives

If Z is simply connected of dimension at least 6, and $n_1 = 1$, $\ell > 2$ then:

$$Z_+ = \coprod_{i=3}^{\ell+1} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \coprod \coprod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2}) \\ \coprod (\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_{\ell+2}-1} \setminus \mathbb{D}^{n-3}).$$

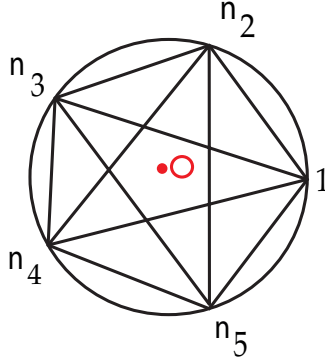
When $n_1 = 1$ and $\ell = 2$ we have the additional complication that making $x_1 = 0$ we pass from the *pentagonal* Z_+ to the *triangular* Z_0 , which is not a connected sum but a product of three spheres and, furthermore, not all its homology below the middle dimension is spherical. All we can say from the above is that

$$Z_+ = (\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_4-1} \setminus \mathbb{D}^{n-3}) \sharp \tilde{\mathcal{E}}$$

where the latter manifold carries all the homology coming from the boundary Z_0 .

We need to be more precise:

The homology generators in Z_+ are those corresponding to $n_2 + n_3$, $n_3 + n_4$, $n_4 + n_5$, $n_2 + n_3 + n_4$ and $n_3 + n_4 + n_5$. Out of these $n_2 + n_3$ and $n_4 + n_5$ give us the handle which is not in $\tilde{\mathcal{E}}$, so the classes in this part are $n_3 + n_4$, $n_2 + n_3 + n_4$ and $n_3 + n_4 + n_5$ all coming from its boundary, which is $Z_0 = (\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_3+n_4-1} \times \mathbb{S}^{n_5-1})$. They can all be killed by taking the union Σ of $\tilde{\mathcal{E}}$ with $W = (\mathbb{S}^{n_2-1} \times \mathbb{D}^{n_3+n_4} \times \mathbb{S}^{n_5-1})$ along their common boundary.



Σ is an $(n - 3)$ -manifold without boundary and one sees from the above that the inclusions of Z_0 in $\tilde{\mathcal{E}}$ and W induce an isomorphism below the top dimension

$$H_*(Z_0) \approx H_*(\tilde{\mathcal{E}}) \bigoplus H_*(W).$$

This implies (by the Mayer-Vietoris sequence) that Σ is a homology sphere. It is simply-connected if Z_+ is so because this implies that $\tilde{\mathcal{E}}$ is so too and that Z_0 is connected.

So Σ is a homotopy sphere and $\tilde{\mathcal{E}}$ is the exterior of W inside a homotopy sphere. Adding the inverse of that homotopy sphere at the interior of $\tilde{\mathcal{E}}$ does not change its diffeomorphism type but turns Σ into a standard sphere. Therefore $\tilde{\mathcal{E}}$ is diffeomorphic to the exterior of a $\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_5-1}$ in \mathbb{S}^{n-3} . It can be expected that this embedding can be assumed to be standard, but for the moment we have the following result:

Theorem 13. *Let $k = 2$, and consider the manifold Z corresponding to the cyclic decomposition $n = n_1 + \dots + n_{2\ell+1}$ and the half manifold $Z_+ = Z \cap \{x_1 \geq 0\}$. When $\ell > 1$ assume Z and $Z_0 = Z \cap \{x_1 = 0\}$ are simply connected and the dimension of Z is at least 6. Then Z_+ is diffeomorphic to:*

a) If $\ell = 1$, the product

$$\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_3-1} \times \mathbb{D}^{n_1-1}.$$

b) If $\ell > 1$ and $n_1 > 1$, the connected sum along the boundary of $2\ell + 1$ manifolds:

$$\prod_{i=2}^{\ell+2} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \prod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2}).$$

c) If $n_1 = 1$ and $\ell > 2$, the connected sum along the boundary of 2ℓ manifolds:

$$\prod_{i=3}^{\ell+1} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \prod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2})$$

$$\prod (\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_{\ell+2}-1} \setminus \mathbb{D}^{n-3}).$$

d) If $n_1 = 1$ and $\ell = 2$, a connected sum along the boundary of two manifolds:

$$(\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_4-1} \setminus \mathbb{D}^{n-3}) \prod \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the exterior of an embedded $\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_5-1}$ in \mathbb{S}^{n-3} .

This theorem clearly implies the following result for the topology of the page of Z' , which includes 3:

Theorem 14. *Let $k = 2$, and consider the manifold Z corresponding to the cyclic partition $n = n_1 + \dots + n_{2\ell+1}$. When $\ell > 1$ assume Z and $Z_0 = Z \cap \{x_1 = 0\}$ are simply connected and the dimension of Z is at least 6. Consider the open book decomposition of Z' given by theorem 1. Then the leaf of this decomposition is diffeomorphic to the interior of:*

a) If $\ell = 1$, the product

$$\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_3-1} \times \mathbb{D}^{n_1-1}.$$

b) If $\ell > 1$ and $n_1 > 1$, the connected sum along the boundary of $\ell + 1$ manifolds:

$$\prod_{i=2}^{\ell+2} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \prod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2}).$$

c) If $n_1 = 1$ and $\ell > 2$, the connected sum along the boundary of 2ℓ manifolds:

$$\prod_{i=3}^{\ell+1} (\mathbb{S}^{d_i-1} \times \mathbb{D}^{n-d_i-2}) \prod_{i=\ell+3}^1 (\mathbb{D}^{d_i-1} \times \mathbb{S}^{n-d_i-2})$$

$$\prod (\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_{\ell+2}-1} \setminus \mathbb{D}^{n-3}).$$

d) If $n_1 = 1$ and $\ell = 2$, a connected sum along the boundary of two manifolds:

$$(\mathbb{S}^{d_2-1} \times \mathbb{S}^{d_4-1} \setminus \mathbb{D}^{n-3}) \amalg \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the exterior of an embedded $\mathbb{S}^{n_2-1} \times \mathbb{S}^{n_5-1}$ in \mathbb{S}^{n-3} .

Acknowledgments. We would like to thank professors Dishant Pancholi and Steven J. Altschuler for very useful comments.

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